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Differential Transform Approaches to Solve One Dimensional Oil-Water Flow Problem in Homogeneous Porous Medium

Aruna Sharma[a*], Amit K Parikh[b]

[a]*Department of Mathematics, Mehsana Urban Institute of Sciences, Ganpat University, Kherva-384012, Mehsana,Gujarat, India.*

[b]*Director Academics, Banaskantha District Kelavani Mandal,G D Modi Vidya Sankul, Palanpur - 385001, Gujarat, India*

Abstract

The current work investigates the instability (fingering) phenomenon in an immiscible flow of oil and water through a homogeneous porous medium. When residual oil is recovered by injecting water into an oil-formatted zone in the secondary oil recovery process, instability phenomena are most commonly observed. The injected water shoots at great speed through the porous material in the shape of fingers, which are unstable due to the injecting force. The partial differential equation that governs this phenomenon is non-linear and obtaining exact solution is sometimes difficult. It is proposed to solve this equation by using two Differential Transform approaches, namely Reduced Differential Transform Method (RDTM) and Hybrid Differential Transform Finite Difference Method (HDTFDM). The solution represents saturation of injected water occupied by schematic fingers, which is obtained in the form of an infinite series. To obtain the numerical solution, a simple iterative strategy is used, which reduces computing time as compared to the traditional Differential Transform Method. The numerical solution of the governing equation and the graphical representation have been obtained using MATLAB. The results obtained by both methods are compared and analysed.

Keywords:- *Double Phase Flow, Hybrid Differential Transform Finite Difference Method (HDFTDM), Instability Phenomenon Reduced Differential Transform Method (RDTM).*

1. Introduction

The main aim of this paper is to analyse the instability phenomenon which takes place during the secondary oil recovery process in a homogeneous porous media.

The problem of the oil-water flow is crucial to the process of recovering oil. 10% to 15% of the oil is collected during the primary oil recovery process, which mostly depends on the pressure which is natural and existing in the oil reservoir. By injecting fluids with lower viscosities, such as water, the leftover oil in the porous oil-formatted media is recovered. This is called the secondary oil recovery technique. These protuberances are referred to as fingers, and this occurrence is brought on by water in the secondary oil recovery process.

When a native fluid (in this case, oil) in an oil-formatted region is displaced by an injected fluid (water) with a lower viscosity, protuberances arise and move through the porous medium at considerably high speeds, instead of the whole front being displaced. These protuberances are referred to as fingers, and this occurrence which is brought on by water being injected is called the instability or fingering phenomenon. The injected fluids that are more transportable than the native fluids could be the cause of instability. The difference in the viscosities of the two immiscible fluids causes the occurrence of these fingers leading to the fingering phenomenon. This phenomenon has major importance in petroleum technology.

Many Researchers have solved this problem under different situations. The statistical behaviour of the instability phenomenon was analysed by '(Scheiddeger ,1960)' with capillary pressure and pressure dependent densities. '(Scheiddeger & Johnson ,1961)'discussed statistical behaviour of the instability phenomenon without capillary pressure. '(Verma,1969)' used the perturbation method to analyse this problem. '(Mehta M N and et al.,2009)'used the group invariant method. '(Meher R and et al., 2010)' studied this problem using exponential self-similar solution techniques. '(Patel K and et al.,2011)'

*Corresponding Author
E-mail address: sharmaaruna961@gmail.com

used the power series method.' (Borana R and et al.,2014)' used the Crank Nicolson Finite Difference Method. '(Piyush Mistry et al.,2013)' used the variational iteration technique. '(Parikh A,2017)' studied this phenomenon in the vertically downward direction.

The partial differential equation that results from the mathematical formulation of this instability phenomenon is non-linear and two Differential Transform Approaches namely Reduced Differential Transform Method (RDTM) '(Keskin, Y., & et al.,2008)' and Hybrid Differential Transform Finite Difference Method (HDTFDM) '(Sungu, Demir, H, 2012 , Yu, L.T., Chen, C.K, 1999)' are used to find the solution of this phenomenon.

2 MATHEMATICAL FORMULATION

A finite cylindrical piece of homogeneous porous media of length 'L' which is fully saturated with native fluid such as oil (o) is under study. Three sides of the cylinder are impermeable and one end is open through which the water which is injected enters the porous medium. This is labelled as the surface (x=0)We consider the cross- sectional area of this cylindrical piece for mathematical study. The difference in viscosity causes the formation of protuberances or fingers, causing the injected water to push the oil out of the oil-formatted area. This gives rise to fingers that are well-developed as shown in Fig (1)

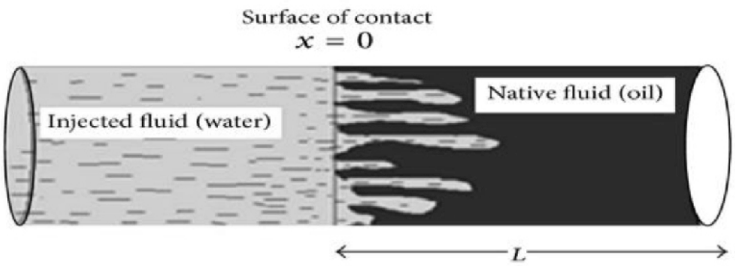

Fig (1)
Source: '(Piyush Mistry and el,2013)'

The length $'x'$ of fingers is taken along the horizontal direction. The irregular fingers are replaced with schematic rectangular fingers.Consider the average cross-sectional area occupied by the schematic fingers as saturation of the injected water at a distance 'x' and time $'t' \geq 0$' for mathematical analysis as shown in Fig(2)

Fig (2)
Source: '(Piyush Mistry and el,2013)'

$S_w(x, t)$ is a function of distance 'x' and time't'. At the common surface $x = 0$,the oil is displaced in the porous medium due to water being injected.

The seepage velocities of the injected water (V_w) and native oil (V_o) are expressed as per '(Scheiddeger ,1960)'

$$V_w = -\frac{k_w}{\mu_w} k \frac{\partial P_w}{\partial x} \tag{1}$$

$$V_o = -\frac{k_o}{\mu_o} k \frac{\partial P_o}{\partial x} \tag{2}$$

The equations of continuity satisfied by the injected water and native oil are

$$\emptyset \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \tag{3}$$

$$\emptyset \frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0 \tag{4}$$

Substituting V_w and V_o from equation (1) and (2) in equation (3) and (4) we have

$$\emptyset \frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left(-\frac{k_w}{\mu_w} k \frac{\partial P_w}{\partial x}\right) = 0 \tag{5}$$

$$\emptyset \frac{\partial S_o}{\partial t} + \frac{\partial}{\partial x}\left(-\frac{k_o}{\mu_o} k \frac{\partial P_o}{\partial x}\right) = 0 \tag{6}$$

where,

k	permeability in m^2
k_w	relative permeability of water
k_o	relative permeability of oil
L	length (m)
p_w	pressure of water
p_o	pressure of oil
V_w	Water seepage velocity (m /s)
V_o	Oil seepage velocity (m /s)
μ_w	constant kinematic viscosity of water
μ_o	constant kinematic viscosity of oil
$\emptyset$	porosity

k_w and k_o are functions of water and oil saturation.

Porous medium is assumed to be fully saturated. The capillary pressure P_c, the pressure difference between the injected water and the native oil, causes the injected water to flow '(Graham, J. W et al., 1959)'.

$$P_c = P_o - P_w \tag{7}$$

Substituting in equation (5), we get

$$\emptyset \frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left(-\frac{k_w}{\mu_w} k \frac{\partial}{\partial x}(P_o - P_c)\right) = 0 \tag{8}$$

Simplifying,

$$\emptyset \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k_w}{\mu_w} k \frac{\partial}{\partial x}(P_o - P_c)\right) \tag{9}$$

$$\emptyset \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k_w}{\mu_w} k \frac{\partial P_0}{\partial x} - \frac{k_w}{\mu_w} k \frac{\partial P_c}{\partial x}\right) \tag{10}$$

Using the result for phase saturation,

$$S_w + S_o = 1 \tag{11}$$

From (11) and (6)

$$\emptyset \frac{\partial}{\partial t}(1 - S_w) + \frac{\partial}{\partial x}\left(-\frac{k_o}{\mu_o} k \frac{\partial P_o}{\partial x}\right) = 0 \tag{12}$$

Simplifying,

$$\emptyset \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left(-\frac{k_o}{\mu_o} k \frac{\partial P_o}{\partial x}\right) \tag{13}$$

From (10) and (13)

$$\frac{\partial}{\partial x}\left(-\frac{k_o}{\mu_o} k \frac{\partial P_o}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{k_w}{\mu_w} k \frac{\partial P_0}{\partial x} - \frac{k_w}{\mu_w} k \frac{\partial P_c}{\partial x}\right) \tag{14}$$

$$\frac{\partial}{\partial x}\left(\frac{k_w}{\mu_w} k \frac{\partial P_c}{\partial x}\right) = \frac{\partial}{\partial x}\left(\left(\frac{k_w}{\mu_w} + \frac{k_o}{\mu_o}\right) k \frac{\partial P_0}{\partial x}\right) \tag{15}$$

Therefore, we get

$$\frac{\partial}{\partial x}\left(\left(\frac{k_w}{\mu_w} + \frac{k_o}{\mu_o}\right) k \frac{\partial P_0}{\partial x} - \frac{k_w}{\mu_w} k \frac{\partial P_c}{\partial x}\right) = 0 \tag{16}$$

Also integrating the above equation, we get,

$$\left(\frac{k_w}{\mu_w} + \frac{k_o}{\mu_o}\right) k \frac{\partial P_0}{\partial x} - \frac{k_w}{\mu_w} k \frac{\partial P_c}{\partial x} = -A(t) \tag{17}$$

where $A(t)$ is the constant of integration. Simplifying equation (17), we get,

$$\frac{\partial P_0}{\partial x} = \frac{\frac{k_w}{\mu_w} k \frac{\partial P_c}{\partial x}}{\left(\frac{k_w}{\mu_w} + \frac{k_o}{\mu_o}\right) k} - \frac{A(t)}{\left(\frac{k_w}{\mu_w} + \frac{k_o}{\mu_o}\right) k} \tag{18}$$

Using (18) in equation (10)

$$\emptyset \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k_w}{\mu_w} k \left[\frac{\frac{k_w}{\mu_w} k \frac{\partial P_c}{\partial x}}{\left(\frac{k_w}{\mu_w} + \frac{k_o}{\mu_o}\right) k} - \frac{A(t)}{\left(\frac{k_w}{\mu_w} + \frac{k_o}{\mu_o}\right) k}\right] - \frac{k_w}{\mu_w} k \frac{\partial P_c}{\partial x}\right) \tag{19}$$

Simplifying we get,

$$\emptyset \frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left(\frac{k_o}{\mu_o} k \frac{\partial P_c}{\partial x} \frac{1}{\left(1 + \frac{k_o \mu_w}{\mu_o k_w}\right)} + \frac{A(t)}{\left(1 + \frac{k_o \mu_w}{\mu_o k_w}\right)}\right) = 0 \tag{20}$$

The relationship between phase saturation and relative permeability [2]

$$k_w = S_w \tag{21}$$

$$k_o = S_o = 1 - S_w$$

The capillary pressure P_c is assumed as a linear function of S_w [6]

$$P_c = -\beta S_w \tag{22}$$

where β is a positive constant. The value of pressure of oil P_o is given by,

$$P_o = \frac{P_o + P_w}{2} + \frac{P_o - P_w}{2} \tag{23}$$

Therefore,

$$P_o = \bar{P} + \frac{P_c}{2} \tag{24}$$

where, $\overline{P}$ = mean pressure and is regarded as constant.

Therefore,

$$\frac{\partial P_o}{\partial x} = \frac{1}{2}\frac{\partial P_c}{\partial x} \tag{25}$$

Using (25) in (17) and using $A(t)$ in (20) we get,

$$\emptyset \frac{\partial S_w}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{k_w}{\mu_w} k \frac{\partial P_c}{\partial S_w}\frac{\partial S_w}{\partial x}\right) = 0 \tag{26}$$

Using,

$$k_w = S_w$$

$$k_o = S_o = 1 - S_w$$

We get,

$$\emptyset \frac{\partial S_w}{\partial t} - \frac{\beta}{2}\frac{k}{\mu_w}\frac{\partial}{\partial x}\left(S_w \frac{\partial S_w}{\partial x}\right) = 0 \tag{27}$$

Therefore,

$$\emptyset \frac{\partial S_w}{\partial t} = \frac{\beta}{2}\frac{k}{\mu_w}\frac{\partial}{\partial x}\left(S_w \frac{\partial S_w}{\partial x}\right) \tag{28}$$

To make equation (28) dimensionless, we use,

$$X = \frac{x}{L}$$

$$T = \frac{k\beta t}{2\mu_w L^2 \emptyset}$$

Where $0 \le X \le 1$ and $0 \le T \le 1$.Equation (28) reduces to,

$$\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial X}\left(S_w \frac{\partial S_w}{\partial X}\right) \tag{29}$$

The initial condition is taken as linear function of X ,

$$S_w(X, 0) = X, \quad 0 \le X \le 1$$

Boundary conditions

$$S_w(0, T) = T, \quad 0 < T \le 1$$

$$S_w(1, T) = 1, \quad 0 \le T < 1$$

3 PROBLEM SOLUTION

Consider equation (29) for the instability phenomenon along with the initial and boundary conditions. This equation will be solved by using two Differential Transform approaches namely

1) Reduced Differential transform Method (RDTM)
2) Hybrid Differential Transform Finite Difference Method (HDTFDM)

3.1 Methodology

3.1.1 *Solution by Reduced Differential Transform Method (RDTM)*

In 2009, Turkish mathematicians Keskin and Oturanc proposed the Reduced Differential Transform Method (RDTM).

Definition 1.

.Let $u(x,t)$ be an analytic and differentiable function w.r.t time 't' and 'x'

Then

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{(t=0)} \tag{30}$$

where $U_k(x)$ is the transformed function of $u(x,t)$.

The original function is denoted by the lower case $u(x,t)$, whereas upper case $U_k(x)$ represents the transformed function.

Definition 2.

The inverse differential transforms of $U_k(x)$ is defined as follows

$$u(x,t) = \sum_{k=0}^{k=\infty} U_k(x)\, t^k \tag{31}$$

Combining expression (30) and (31), we get,

$$u(x,t) = \sum_{k=0}^{k=\infty} \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{(t=0)} t^k \tag{32}$$

The basic concept of RDTM is based on power series expansion.

Table 1.

Table of Fundamental Operations

Original function	**Transformed Function**
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{(t=0)}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \boldsymbol{c}u(x,t)$	$W_k(x) = c\ U_k(x), c\ constant$
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n)$, $where\ \delta(k-n) = \begin{cases} 1, & k=n \\ 0, & k \neq n \end{cases}$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{r=k} U_r(x)\ V_{k-r}(x)$
$w(x,t) = \frac{\partial^r u(x,t)}{\partial t^r}$	$W_k(x) = \frac{(k+r)!}{k!}\ U_{k+r}(x)$
$w(x,t) = \frac{\partial u(x,t)}{\partial x}$	$W_k(x) = \frac{\partial}{\partial x} U_k(x)$
$w(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$	$W_k(x) = \frac{\partial^2}{\partial x^2} U_k(x)$

The non-linear instability phenomenon governed by equation (29) which is reduced to

$$\frac{\partial S_w}{\partial T} = S_w \frac{\partial^2 S_w}{\partial X^2} + \left(\frac{\partial S_w}{\partial X}\right)^2 \tag{33}$$

will be solved using RDTM

Here,

$$S_w(X,T) = \sum_{k=0}^{k=\infty} S_k(X)\, T^k \tag{34}$$

Here,

$$S_k(x) = \frac{1}{k!}\left[\frac{\partial^k S_w(X.T)}{\partial T^k}\right]_{(T=0)} \tag{35}$$

Therefore,

$$S_w(X,T) = \sum_{k=0}^{k=\infty} \frac{1}{k!}\left[\frac{\partial^k S_w(X.T)}{\partial T^k}\right]_{(T=0)} T^k \tag{36}$$

Applying the Fundamental operations from Table 2

$$\frac{\partial S_w}{\partial T} = (k+1)\ \ S_{k+1}(X) \tag{37}$$

$$S_w \frac{\partial^2 S_w}{\partial X^2} = \sum_{r=0}^{r=k} \left(S_r(X)\right) \frac{\partial^2}{\partial X^2} S_{k-r}(X) \tag{38}$$

$$\left(\frac{\partial S_w}{\partial X}\right)^2 = \sum_{r=0}^{r=k} \frac{\partial}{\partial X}\left(S_r(X)\right) \frac{\partial}{\partial X}\left(S_{k-r}(X)\right) \tag{39}$$

Transforming the initial conditions to,

$$S_0(X) = X,\ \ 0 \le X \le 1 \tag{40}$$

Using (37), (38) and (39), equation (33) is transformed to**,**

$$(k+1)\ \ S_{k+1}(X) = \sum_{r=0}^{r=k} \left(S_r(X)\right) \frac{\partial^2}{\partial X^2} S_{k-r}(X) + \sum_{r=0}^{r=k} \frac{\partial}{\partial X}\left(S_r(X)\right) \frac{\partial}{\partial X}\left(S_{k-r}(X)\right) \tag{41}$$

Equation (41) is the recurrence relation, in which putting $k = 0,1,2,3.$ the coefficients $S_1(X), S_2(X), S_3(x) \dots \dots \dots are\ obtained$.

These are used in Equation (34) to obtain

$$S_w(X.T) = \ \ S_0(X) + S_1(X)T + S_2(X)T^2 + S_3(X)T^3 + S_4(X)T^4 \dots \dots \tag{42}$$

Therefore for $k = 0, from\ recurrence\ relation$ (41),we get

$$S_1(X) = S_0(X)\frac{\partial^2}{\partial X^2} S_0(X) + \frac{\partial}{\partial X}\ \ S_0(X)\frac{\partial}{\partial X}\ \ S_0(X) \tag{43}$$

$S_2(x)$=0

$S_3(x)$=0 and so on.

Therefore, from equation (42) we get,

$$S_w(X.T) = \ \ X + T \tag{44}$$

Equation (44) gives the series solution for nonlinear equation using RDTM

Numerical representation of the solution

The numerical values of the saturation obtained from equation (44) for different distances $'X'$ at $\mathrm{T} = 0.001, 0.002, 0.003, 0.004, 0.005, 0.006, 0.007, 0.008, 0.009, .01$ are obtained by using MATLAB and presented in **Table 2** below.

Table 2: Saturation Vs. Distance X at fixed time T

Saturation $S_w(X,T)$ for values of distance' X ' and time ' T ' by RDTM

X\T	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.01
0	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.01
0.1	0.101	0.102	0.103	0.104	0.105	0.106	0.107	0.108	0.109	0.11
0.2	0.201	0.202	0.203	0.204	0.205	0.206	0.207	0.208	0.209	0.21
0.3	0.301	0.302	0.303	0.304	0.305	0.306	0.307	0.308	0.309	0.31
0.4	0.401	0.402	0.403	0.404	0.405	0.406	0.407	0.408	0.409	0.41
0.5	0.501	0.502	0.503	0.504	0.505	0.506	0.507	0.508	0.509	0.51
0.6	0.601	0.602	0.603	0.604	0.605	0.606	0.607	0.608	0.609	0.61
0.7	0.701	0.702	0.703	0.704	0.705	0.706	0.707	0.708	0.709	0.71
0.8	0.801	0.802	0.803	0.804	0.805	0.806	0.807	0.808	0.809	0.81
0.9	0.901	0.902	0.903	0.904	0.905	0.906	0.907	0.908	0.909	0.91
1	1	1	1	1	1	1	1	1	1	1

From the table. we observe that $\boldsymbol{S_w(X,T)}$ which is the saturation of water injected is increasing when distance ‘ X ‘is increasing for fixed time ‘ T’. It is also increasing when $'T'$ is increasing for fixed distance $'X'$

Figure 3 shows a 3D plot of saturation $\boldsymbol{S_w(X.T)}$ against distance ‘ X ' for T = 0.001, 0.002, 0.003, 0.004, 0.005, 0.006, 0.007, 0.008, 0.009, .01 by RDTM

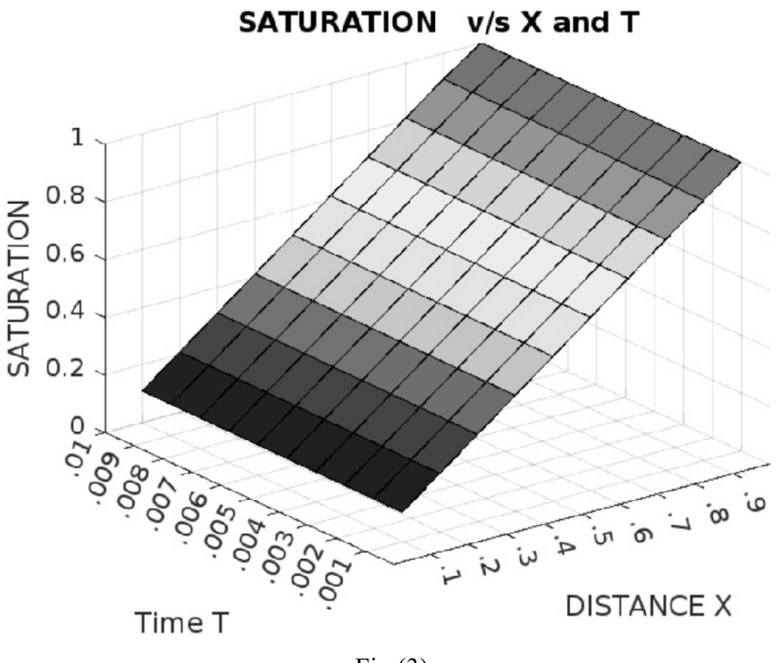

Fig (3)

From **Figure 3,** it can be concluded that for any fixed time ‘ T ‘, as the distance ‘X’ increases the saturation $\boldsymbol{S_w(X.T)}$ of injected water increases linearly.

3.1.2 ***Solution by Hybrid Differential Transform Finite Difference Method (HDTFDM)***

The Hybrid Differential Transform and Finite Difference Method (HDTFDM) is used to solve the non-linear partial differential equation [29]. The Differential Transform Method (DTM) and the Finite Difference Method (FDM) are combined in this method.

The spatial variables are approximated using the FDM, while the time variable is approximated using the DTM.

'(Zhou,1986)' proposed the DTM using which he differential equations both linear and non-linear in electric circuit analysis. '(Yu, L.T., Chen, C.K, 1999)' used the hybrid method.

For the 'T' variable, the differential transform is used, and on the 'X variable, the FDM is used. The hybrid approach can solve both linear and nonlinear partial differential equations because it converges quickly after a few iterations.

3.1.3 ***Preliminaries***

The Differential Transform of the k^{th} derivative of $u(x,t)$ applied to the 't' variable is given as

$$U(i,k) = \frac{1}{k!}\left[\frac{d^k u(x,t)}{dt^k}\right]_{t=0} \qquad k = 0,1,2,\ldots \text{ and } i = 0,1,2 \tag{45}$$

The inverse Transform of $U(i,k)$ is given as

$$u(x,t) = \sum_{k=0}^{\infty} U(i,k)\, t^k \tag{46}$$

where $u(x,t)$ in lower case letters represents the original function and $U(i,k)$ in upper case letters represents transformed function.

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k u(x,t)}{dt^k}\right]_{t=0} t^k \tag{47}$$

where $U(i,k) = U(x_i, k)$, $x_i = ih$, $i = 0,1,2,3,\ldots..$

The step interval is denoted by h for FDM.

The theorems stated below follow from '(Derya, A., 2020)'

Theorem 1. $If\ f(x,t) = \frac{\partial m}{\partial t}, then\ F(i,k) = (k+1)M(i,k+1)$

Theorem 2. $If\ f(x,t) = \frac{\partial^2 m}{\partial t^2}\ , then\ F(i,k) = (k+1)(k+2)M(i,k+2)$

Theorem 3. $If\ f(x,t) = xe^{-t}\ . then\ F(i,k) = x\frac{(-1)^k}{k!}$

Theorem 4. $If\ f(x,t) = \sin x\ , then\ F(i,k) = \sin x$

Theorem 5. $If\ f(x,t) = \sin t\ , then\ F(i,k) = \sin\left(\frac{\pi k}{2}\right)\frac{1^k}{k!}$

Theorem 6. $If\ f(x,t) = \frac{\partial m}{\partial x}(x,t), then$

$$F(i,k) = \frac{M(i+1,m) - M(i-1,m)}{2h}$$

Theorem 7. $If\ f(x,t) = m(x,t)\frac{\partial m}{\partial x}(x,t), then$

$$F(i,k) = \sum_{m=0}^{k} M\,(i,k-m)\frac{M(i+1,m) - M(i-1,m)}{2h}$$

3.2 NUMERICAL SOLUTION

The Hybrid Differential Transform and Finite Difference Method (HDTFDM) and theorems given above are applied to Equation (33)

$$\frac{\partial S_w}{\partial T} = S_w \frac{\partial^2 S_w}{\partial X^2} + \left(\frac{\partial S_w}{\partial X}\right)^2$$

Initial conditions $S_w(X,0) = X, \quad 0 < X \leq 1$

Boundary conditions $S_w(0.T) = T, \quad 0 < T \leq 1$

$$S_w(1,T) = 1, \quad 0 \leq T < 1$$

Applying, Differential Transformation to the 'T' variable and finite difference to the 'X' variable and using theorems given above, we have,

$$\frac{\partial S_w}{\partial T} = (k+1)\mathbf{S}(i,k+1) \tag{48}$$

$$S_w \frac{\partial^2 S_w}{\partial X^2} = \sum_{r=0}^{k} \mathbf{S}(i,k-r) \frac{\mathbf{S}(i+1,r) - 2\mathbf{S}(i,r) + \mathbf{S}(i-1,r)}{h^2} \tag{49}$$

$$\left(\frac{\partial S_w}{\partial X}\right)^2 = \sum_{r=0}^{k} \frac{\mathbf{S}(i+1,r) - \mathbf{S}(i-1,r)}{2h} \frac{\mathbf{S}(i+1,k-r) - \mathbf{S}(i-1,k-r)}{2h} \tag{50}$$

where $S_w(X,T)$ is the original function and, $\mathbf{S}(i,\mathrm{k}) = \mathbf{S}(X_{i,}, k)\ X_{i,} = ih$, $i = 0,1,2,3, \ldots..$ is transformed function

Transforming, the initial conditions and boundary conditions to,

$$\mathbf{S}(i,0) = \mathbf{S}(X_i,0) = f(X_i)\ , X_i = ih\ , i = 0,1,2,3, \ldots$$

$$\mathbf{S}(0,k) = 1 \text{ , for k} = 1$$
$$= 0 \text{ otherwise}$$
$$\mathbf{S}\ (N,k) = \delta(k) = 1 \text{ for } k = 0$$
$$= 0 \quad k \neq 0$$

where N is the no. of spatial segments.

Substituting in equation (33) we get, according to the hybrid method, the recurrence relation,

$$(\mathrm{k}+1)\mathbf{S}\ (i,k+1) = \sum_{r=0}^{k} \mathbf{S}(i,k-r) \frac{\mathbf{S}(i+1,r) - 2\mathbf{S}(i,r) + \mathbf{S}(i-1,r)}{h^2} + \sum_{r=0}^{k} \frac{\mathbf{S}(i+1,r) - \mathbf{S}(i-1,r)}{2h} \frac{\mathbf{S}(i+1,k-r) - \mathbf{S}(i-1,k-r)}{2h} \tag{51}$$

for $k = 0,1,2,3, \ldots$ we obtain $\mathbf{S}(i,1), \mathbf{S}(i,2), \ldots.$ The approximate solutions for various values 'X' and 'T' are found using the inverse transformation,

$$S_w(X,T) = \sum_{k=0}^{\infty} \mathbf{S}(i,k)\, T^k \tag{52}$$

$when,\quad X_i=0$

$$S_w(X,T)=\sum_{k=0}^{\infty}\mathbf{S}(i,k)\,T^k \tag{53}$$

$$S_w(0,T)=\sum_{k=0}^{\infty}\mathbf{S}(0,k)\,T^k$$

$$=\mathbf{S}(0,0)+\mathbf{S}(0,1)T+\mathbf{S}(0,2)T^2+\cdots$$

$$=\boldsymbol{T}$$

where $X_i = ih$,for $h = 0.1,\ i = 0,1,2,\ldots.$

4 RESULTS AND DISCUSSION

The numerical values of the saturation obtained from equation (52) for various distances ' X ' at fixed time T=0.001,0.002,0.003,0.004,0.005,0.006,0.007,0.008,0.009,.01 are obtained by using MATLAB and presented in **Table 3** below.

Table 3: Saturation S_w (X, T) for 'X' and 'T' by HDTFDM

X/T	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.01
0	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.01
0.1	0.101	0.102	0.103	0.104	0.105	0.106	0.107	0.108	0.109	0.11
0.2	0.201	0.202	0.203	0.204	0.205	0.206	0.207	0.208	0.209	0.21
0.3	0.301	0.302	0.303	0.304	0.305	0.306	0.307	0.308	0.309	0.31
0.4	0.401	0.402	0.403	0.404	0.405	0.406	0.407	0.408	0.409	0.41
0.5	0.501	0.502	0.503	0.504	0.505	0.506	0.507	0.508	0.509	0.51
0.6	0.601	0.602	0.603	0.604	0.605	0.606	0.607	0.608	0.609	0.61
0.7	0.701	0.702	0.703	0.704	0.705	0.706	0.707	0.7079	0.7089	0.7099
0.8	0.801	0.802	0.803	0.8039	0.8049	0.8058	0.8067	0.8076	0.8084	0.8093
0.9	0.901	0.9018	0.9026	0.9034	0.904	0.9047	0.9053	0.9059	0.9064	0.9069
1	1	1	1	1	1	1	1	1	1	1

The table shows that the saturation S_w $(\boldsymbol{X},\boldsymbol{T})$ increases linearly with 'X' for a fixed time 'T'and that saturation likewise increases with time 'T' for a fixed distance 'X'.

Graphical Representation:

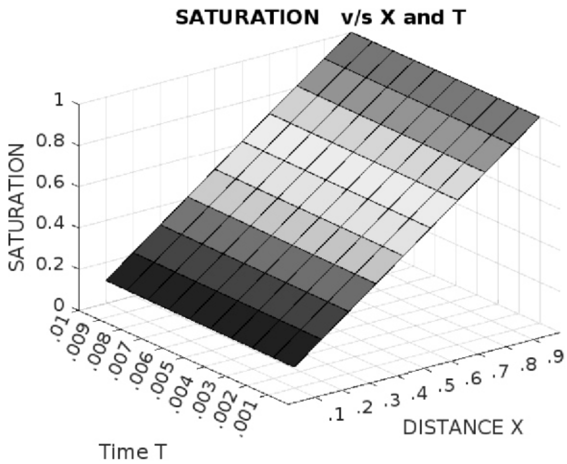

Fig (4)

Fig 4: Saturation S_w (X, T) for 'X' and 'T' by HDTFDM

From the 3D plot in Fig.4, we observe that as 'X' increases for a fixed time T', as well as time 'T' is increasing for a fixed distance 'X', the saturation $S_w(\boldsymbol{X},\boldsymbol{T})$ increases

Table 4. COMPARISON OF RDTM and HDTFDM

XT	0.001		0.003		0.005		0.007		0.009	
	RDTM	HDTFDM	RDTM	HDTFDM	RDTM	HDTFDM	RDTM	HDTFDM	RDTM	HDTFDM
0	0.001	0.001	0.003	0.003	0.005	0.005	0.007	0.007	0.009	0.009
0.1	0.101	0.101	0.103	0.103	0.105	0.105	0.107	0.107	0.109	0.109
0.2	0.201	0.201	0.203	0.203	0.205	0.205	0.207	0.207	0.209	0.209
0.3	0.301	0.301	0.303	0.303	0.305	0.305	0.307	0.307	0.309	0.309
0.4	0.401	0.401	0.403	0.403	0.405	0.405	0.407	0.407	0.409	0.409
0.5	0.501	0.501	0.503	0.503	0.505	0.505	0.507	0.507	0.509	0.509
0.6	0.601	0.601	0.603	0.603	0.605	0.605	0.607	0.607	0.609	0.609
0.7	0.701	0.701	0.703	0.703	0.705	0.705	0.707	0.707	0.709	0.7089
0.8	0.801	0.801	0.803	0.803	0.805	0.8049	0.807	0.8067	0.809	0.8084
0.9	0.901	0.901	0.903	0.9026	0.905	0.904	0.907	0.9053	0.909	0.9064
1	1	1	1	1	1	1	1	1	1	1

It can be observed that the numerical solution obtained by HDTFDM and RDTM closely agree at almost all values of 'X' and 'T'. The values close to the upper and lower boundary are sometimes found to be different for RDTM and slightly higher than HDTFDM. The reason for this being that while we obtain the solution of equation using RDTM the boundary conditions are not used, whereas in HDTFDM the boundary conditions are used while calculating the values.

This is because RDTM is independent of the boundary conditions, because of which the values close to boundary become unstable after sometime.

COMPARISON OF FDM and HDTFDM

X\T	0.001		0.003		0.005	
	FDM	**HDTFDM**	**FDM**	**HDTFDM**	**FDM**	**HDTFDM**
0.1	**0.101**	**0.101**	**0.103**	**0.103**	**0.105**	**0.105**
0.2	**0.201**	**0.201**	**0.203**	**0.203**	**0.205**	**0.205**
0.3	**0.301**	**0.301**	**0.303**	**0.303**	**0.305**	**0.305**
0.4	**0.401**	**0.401**	**0.403**	**0.403**	**0.405**	**0.405**
0.5	**0.501**	**0.501**	**0.503**	**0.503**	**0.505**	**0.505**

It can be observed that the numerical solution obtained by HDTFDM and FDM '(Borana et al.,2014)' closely agree at almost all values of 'X' and 'T'

Convergence criteria for HDTFDM and RDTM

From equation (46) we get the series solution for the nonlinear PDE as

$$u(x,t) = \sum_{k=0}^{\infty} U(i,k)(t-t_0)^k \ \ or \ \ u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^k \qquad \text{(A)}$$

$$where\ , t \in l\ , where\ l = (t_0, t_0 + r), r > 0$$

Here $t_0 = 0$, then the convergence of the power series in 't' can be found as per the following theorem '(Moosavi Noori, S.R. and Taghizadeh, N, 2021)'

Theorem 1.

If $\varphi_k(x,t) = U(i,k)(t-t_0)^k$,then the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$, stated in equation (A),$\forall k \in N \cup \{0\}$ follows the following criteria.

(i) If $\exists\ 0 < \lambda < 1$, such that $\|\varphi_{k+1}\| \leq \lambda\|\varphi_k\|$, series is convergent.

(ii) If $\exists\ \lambda > 1$, such that $\|\varphi_{k+1}\| \geq \lambda\|\varphi_k\|, series$ is divergent

5 CONCLUSIONS

MATLAB is used to obtain the numerical solution and graphs for calculating the saturation of water which is injected in homogeneous porous media for the instability phenomenon with regard to distance and time

The graph given by Figure. 1 and 2 shows that saturation of injected water in the horizontal porous medium increases with increase in distance X for T > 0 . The saturation is found to be linearly increasing with increasing time and it is consistent with physical phenomenon.

Comparative study of obtaining the result by the two methods, namely, HDTFDM and RDTM shows that results closely agree with each other. Except, in RDTM method the values close to the boundary start becoming unstable for higher values of 'T'. The results closely agree with Finite Difference Method '(Borana, R. N et el.,2014)'. In both methods computational difficulties are reduced as compared to other approaches, and all calculations can be performed easily and accurately.

The solution expressed as a series demonstrates that for T >0, we can obtain the approximate value of saturation at any 'X' easily as there is no need of linearisation or perturbation. The series expression in both methods has been used to obtain numerical values emphasizing the ability to obtain analytic and numerical solutions by these two methods. Even, the results are found to be reliable and accurate and the series has been found to converge to exact solutions. As a result, we can use these methods to solve many difficult partial differential equations both linear and non-linear without the need for linearization or perturbation.

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